Comment on "Exact solution of the wave dynamics of a particle bouncing chaotically on a periodically oscillating wall"

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We attempt to show both analytically and numerically that a recent work by Willemsen [Phys. Rev. E **50**, 3116 (1994)] is not correct and the proposed exact solution of the so-called bouncer model appears to be incorrect.

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Recently Willemsen published a paper [1] on the quantum description of a point particle bouncing on a periodically oscillating plate in a uniform gravitational field.

For a number of reasons quantum theories of bouncing ball models have been present in the literature in the last decade. We discuss here problems connected with solving the Schrödinger equation with time-dependent boundary conditions [2–5], fingerprints of chaos in quantum mechanics [6], and many other topics of the so-called quantum chaology.

The latter problems require periodic, at least, time dependencies of the position of the reflecting plate. It was believed that only these periodic time dependencies, which in addition lead to a piecewise separable (in position and time) Schrödinger equation, allow an analytic solution. These are the reasons why the statement of Ref. [1], which announces finding the exact solution of the Schrödinger equation describing a ball bouncing on a harmonically oscillating plate, i.e., solving the problem in the case that excludes a separation of variables, attracted our genuine attention.

The purpose of this Comment, however, is to demonstrate our reasoning that the cited statement of Ref. [1] is false. The author of Ref. [1] seeks a solution of the following equation

$$i\partial_t\Phi = -\tfrac{1}{2}\partial_z^2\Phi + gz\Phi, \quad z \geq x_\omega(t), \tag{1}$$

subject to the boundary conditions

$$\Phi[z = x_\omega(t), t] = 0, \;\; x_\omega(t) = h_0[1 + \cos(\omega t)]. \tag{2}$$

Changing variables $x = z - x_\omega(t)$ and making the replacement $\phi(x,t) = \Phi(x,t)\exp[igh_0(t + \sin(\omega t)/\omega)]$, one gets

$$i\partial_t\phi = [H_0 + f(t)p]\phi, \quad x \geq 0, \tag{3}$$

where

$$H_0 = -\frac{1}{2}\partial_x^2 + gx, \;\; p = -i\partial_x, \;\; f(t) = -\frac{d}{dt}x_\omega(t). \tag{4}$$

The function ϕ fulfills the following time-independent boundary condition $\phi(x = 0, t) = 0$. The solution of Eq. (3) as given in [1], reads

$$\phi(x,t) = e^{-i\xi(t)}\sum_n \phi_n(x)e^{-iE_n t}\int_0^\infty dx'\phi_n(x')\Psi_0[x' - h(t)], \tag{5}$$

where $\xi(t) = -g\int_0^t t'f(t')dt'$, $h(t) = \int_0^t f(t')dt' = x_\omega(0) - x_\omega(t)$ and $H_0\phi_n(x) = E_n\phi_n(x)$, $\phi_n(0) = 0$, and $\Psi_0(x) = \phi(x, t = 0)$.

In the first part of this Comment it will be shown that, contrary to what is claimed in Ref. [1], the formula given in (5) is not a solution of Eq. (3). To this end let us substitute (5) into (3), multiply the result by $\phi_m(x)$, and then integrate from 0 to ∞. Thus, we get

$$f(t)u_m(t) \equiv f(t)\left[-gta_m + i\int_0^\infty \Psi_0[x - h(t)]\partial_x\phi_m(x)dx - \sum_n a_n e^{i(E_m - E_n)t}\langle\phi_m|p|\phi_n\rangle\right] = 0, \tag{6}$$

with

$$a_n = \int_0^\infty dx'\phi_n(x')\Psi_0[x' - h(t)]. \tag{7}$$

Note that the eigenfunctions $\phi_n(x)$ form the complete orthonormal set of functions of the Sturm-Liouville problem posed by H_0 and the boundary conditions. Assuming that $\Psi_0(x) = \Theta(x)\phi_0(x)$, with $\Theta(x)$ being the standard step function, we will now prove that apart from the discrete set of values (t_k) of the variable t, $f(t_k) = 0$, Eq. (6) is not satisfied. To this end, note that the equation $\mathrm{Re}[u_m(t)] = 0$ now reads

$$0 = -gtI_{0,m}(t) + g\chi_{0,m}(t) \tag{8}$$

with

$$I_{0,m}(t) = \int_{h(t)}^{\infty} \phi_0[x - h(t)]\phi_m(x)dx,$$
$$\chi_{0,m} = \sum_{n,n\neq m} \frac{\sin(E_m - E_n)t}{E_m - E_n} I_{0,n}(t) \quad (9)$$

and we have used the exact relation [7]: $(E_m - E_n)\langle\phi_m|p|\phi_n\rangle = ig(\delta_{m,n} - 1)$. The integral in (9) can be performed analytically [8] for the Airy functions $\phi_n(x) = c_n\mathrm{Ai}(\alpha x - z_n)$ with constant α, $c_n = \alpha^{1/2}/\mathrm{Ai}'(-z_n)$ and $-z_n$ being zeroes of $\mathrm{Ai}(y)$, $E_n = \alpha^2 z_n/2$. The result reads

$$I_{0,n}(t) = c_0c_n \int_h^{\infty} \mathrm{Ai}(\alpha(x-h) - z_0)\mathrm{Ai}(\alpha x - z_n)dx = \frac{c_0c_n}{\alpha(z_0 + \alpha h - z_n)}\mathrm{Ai}'(-z_0)\mathrm{Ai}(\alpha h - z_n), \quad (10)$$

with $\mathrm{Ai}'(x)$ being a derivative of the Airy function. For large m the inequality $|I_{0,m}| < N/|z_m - z_0 - \alpha h|$ holds with N being a constant. The proof that $\chi_{0,m}(t)$ is bounded goes as follows:

$$|\chi_{0,m}(t)| < \sum_{n,n\neq m} \left|\frac{\sin[(E_m - E_n)t]}{E_m - E_n} I_{0,n}(t)\right| < \sum_{n,n\neq m} \frac{1}{|E_m - E_n|}\frac{N}{|z_n - z_0 - \alpha h|}.$$

For large n ($n \gg m$) terms in the above series behave as $1/E_n^2$, i.e., $n^{-4/3}$, which means that the series is convergent and $\chi_{0,m}(t)$ is bounded. Going back to Eq. (8) and recalling that $I_{0,m}(t)$ is a bounded oscillating function of t, it is immediately seen that the right-hand side of (8) diverges as $t \to \infty$ and thus Eq. (8) does not hold.

Bearing this fact in mind one shall not be surprised by the obviously incorrect consequences that stem from the alleged solution given in Eq. (25) of [1], i.e., our Eq. (5). Note first of all that for $t = T = 2\pi/\omega$ and $\Psi_0(x) = \phi_k(x)$ it follows from (5), due to $h(T) = 0$, that

$$\phi(x,T) = e^{-i\xi(T)}e^{-iE_kT}\phi_k(x) \quad (11)$$

and this means in turn that the operator U, which transforms the state from $t = 0$ to $t = T$, is diagonal in the H_0 representation.

In what follows we present results of the numerical calculations of the matrix elements of U and demonstrate that U is not diagonal. If in (3) a change of the phase of the wave function is performed $\phi(x,t) = \exp\{i[-fx + (1/2)\int_0^t f^2(t')dt']\}\Gamma(x,t)$, then (3) reads

$$i\partial_t\Gamma(x,t) = [-\tfrac{1}{2}\partial_x^2 + (g - \dot{f})x]\Gamma(x,t). \quad (12)$$

TABLE I. Probabilities $P_n(T)$ as calculated from Eq. (14) for $h_0 = 0.25, g = 1.0$, and $\omega = 1.0$. N = number of basis functions.

n	N=10	N=20	N=100	N=400
1	0.8986455	0.8986415	0.8986411	0.8986411
2	0.0916495	0.0916527	0.0916531	0.0916531
3	0.0090242	0.0090246	0.0090246	0.0090246
4	0.0006482	0.0006482	0.0006482	0.0006482
5	0.0000161	0.0000160	0.0000160	0.0000160
6	0.0000118	0.0000118	0.0000118	0.0000118
7	0.0000039	0.0000036	0.0000036	0.0000036
8	0.0000005	0.0000005	0.0000005	0.0000005
9	0.0000001	0.0000001	0.0000001	0.0000001
10	0.0000002	0.0000004	0.0000004	0.0000004

Writing the function Γ in the form

$$\Gamma(x,t) = \sum_n b_n(t)\phi_n(x) \quad (13)$$

one obtains for the b_n the following system of ordinary differential equations:

$$i\dot{b}_m = E_m(1 - 2\dot{f}/3g)b_m + \dot{f}g \sum_{n,n\neq m} \frac{b_n}{(E_n - E_m)^2}. \quad (14)$$

In Eq. (14) we used matrix elements of x as given in Ref. [8]. We integrated numerically the above equations for $b_n(t = 0) = \delta_{n,k}$ with $k = 1$ and in Table I the probabilities $P_n(T) = |U_{n,k}|^2 = |b_n(T)|^2$ are presented. Results of the formalism developed in Ref. [1] would give in this case $P_n(T) = |a_n(T)|^2 = \delta_{n,k} = |a_n(0)|^2 = P_n(0)$. It is needless to stress the fundamental differences in the behavior of the considered model in both cases (e.g., the lack or occurrence of diffusion, respectively).

In conclusion let us state that we feel that we have demonstrated both analytically and numerically that the solution proposed in [1] is not correct. The fact that formal solutions of the Schrödinger equation are valid only for a very restricted class of functions is not always remembered and often leads to erroneous conclusions as was noted, e.g., in [3,9,10].

The criticized work and those cited above raise very subtle and difficult questions on working in a half space in quantum mechanics. The problems of a similar nature are also encountered in the case of the celebrated model of a particle in the infinitely deep potential well with time-dependent width [11,12]. Unfortunately, the possible sources of troubles concerning problems in restricted spaces are far from being understood in the existing literature.

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